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Coherent states for systems with discrete and continuous spectrum

Jean Pierre Gazeau†§ and John R Klauder‡‖

† Laboratoire de Physique Théorique de la Matière Condensée, Université Paris 7—Denis-Diderot, 2 Place Jussieu, 75251 Paris Cedex 05, France
‡ Departments of Physics and Mathematics, University of Florida, Gainesville, FL 32611, USA

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Abstract. For a system with one degree of freedom, coherent states that are parametrized by classical canonical action–angle variables are introduced. These states also possess continuity of labelling, a resolution of unity, and temporal stability. The insistence on canonical action–angle variables strongly restricts any remaining arbitrariness in the coherent state definition. Such states are introduced for semibounded Hamiltonian operators having either a discrete or a continuous spectrum. Hamiltonians that have both discrete and continuous parts in their spectrum are also discussed.

1. Introduction

Coherent states are generally acknowledged to provide a close connection between classical and quantum formulations of a given system. For convenience at this point, we consider only a real two-parameter set of coherent states, say $\{|J,\gamma\rangle\}$, $J \geqslant 0$, and $-\infty < \gamma < \infty$. A suitable set of requirements for these states is given, in association with a specific Hamiltonian operator $\mathcal{H}$, by

(a) Continuity: $(J',\gamma') \longrightarrow (J,\gamma) \Rightarrow |J',\gamma'\rangle \longrightarrow |J,\gamma\rangle$.
(b) Resolution of unity: $\mathbb{1} = \int |J,\gamma\rangle\langle J,\gamma|\,\mathrm{d}\mu(J,\gamma)$.
(c) Temporal stability: $\mathrm{e}^{-\mathrm{i}\mathcal{H}t}|J,\gamma\rangle = |J,\gamma+\omega t\rangle$, $\omega = \text{constant}$.
(d) Action identity: $\langle J,\gamma|\mathcal{H}|J,\gamma\rangle = \omega J$.

The first two requirements are standard, emphasizing the fact that the identity operator may be understood in a restricted sense, namely as a projector onto a finite or infinite subspace. The third requirement ensures that the time evolution of any coherent state is always a coherent state. Observe, in this evolution, J remains constant while γ increases linearly. These properties are similar to the classical behaviour of action–angle variables. If J and γ denote canonical action–angle variables, they would enter the classical action in the form

$$I = \int_0^T (J\dot{\gamma} - \omega J)\,\mathrm{d}t. \tag{1}$$

§ E-mail address: gazeau@ccr.jussieu.fr
‖ E-mail address: klauder@phys.ufl.edu

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In fact, the classical action functional can be viewed as the restricted evaluation of the quantum action functional, namely

$$I = \int_0^T \left[\mathrm{i}\langle J, \gamma| \frac{\mathrm{d}}{\mathrm{d}t} |J, \gamma\rangle - \langle J, \gamma|\mathcal{H}|J, \gamma\rangle \right] \mathrm{d}t \tag{2}$$

for different paths $\{|J(t), \gamma(t)\rangle \,:\, 0 \leqslant t \leqslant T\}$ lying in a two-dimensional manifold in Hilbert space. Thus the fourth requirement simply codifies the fact that the two coordinates (J, γ) are canonical action–angle variables (it will follow that the kinematical term is $J\dot{\gamma}$ as needed).

Stationary variations of the classical action functional lead to the equations of motion $\dot{\gamma} = \omega$ and $\dot{J} = 0$ with solutions $\gamma(t) = \gamma + \omega t$ and $J(t) = J$ expressed in terms of their initial values at $t = 0$. Normally, the states $|J, \gamma + \omega t\rangle$ would be an approximation to the true quantum temporal evolution, but the third requirement asserts that the path in Hilbert space represented by $\{|J, \gamma + \omega t\rangle \,:\, 0 \leqslant t \leqslant T\}$ is actually the true quantum temporal evolution for the quantum Hamiltonian $\mathcal{H}$. Thus, the restricted quantum action functional in this case is *exact* [1]; a wider set of variational paths that all start at $|J, \gamma\rangle$ at $t = 0$ inevitably leads to the same extreme path. As we shall see, we are able to find coherent states $\{|J, \gamma\rangle\}$ that satisfy the four requirements for a large class of systems with Hamiltonian operators, $\mathcal{H}$, having either a discrete or continuous spectrum. In a certain sense, we can also deal with a Hamiltonian having both a discrete and continuous spectrum, and in this case, for convenience, we do not impose the fourth requirement. Of course, temporal stability and action identity are minimal requirements which may need to be weakened when dealing with different parts of the spectral resolution of $\mathcal{H}$, as it could be for instance for multiple-band hamiltonians. We shall encounter an example of such an adaptation in section 3, where we modify the action parameter of the coherent state, $J \to s(J)$, and we could do the same for the time evolution of the 'angle', $\gamma + \omega t \to \gamma(t)$ if necessary.

In earlier work, one of the authors (JRK) discussed systems of coherent states characterized by only the first three requirements given above [2–4]. It was found that many distinct coherent-state families could be found for any given Hamiltonian with discrete spectrum. It is noteworthy that the addition of the fourth requirement selects one coherent-state family from the many possible ones, up to a possible remaining freedom in the choice of measure $\mathrm{d}\mu(J, \gamma)$ resulting from a classical moment problem.

It should be noted that Nieto and co-workers [5] have also studied coherent states for very general potentials. However, their results have validity only to the lowest $\hbar$ dependence, *i.e.*, they are semi-classical in character. By contrast, the coherent states presented in this paper lead to canonical action–angle variables that are valid for *all* values, namely, small values deep in the quantum region as well as large values. We also note that coherent states for a continuous spectrum have been previously considered; see, e.g., [6].

In section 2 we study the case for a discrete spectrum. In section 3, we deal with the case of a continuous spectrum. Finally, in section 4 we examine systems with both a discrete spectrum and continuous spectrum. The present work is motivated by potential applications to the hydrogen atom and to more general atomic systems for which experimental elaboration of such coherent states can be conceived as attainable; however such questions are not discussed in this paper. We generally use units in which $\hbar = 1$.

2. Coherent states for discrete dynamics

Choose a Hamiltonian $\mathcal{H}$ with a discrete spectrum which is bounded below and has been adjusted so that $\mathcal{H} \geqslant 0$. For convenience in our presentation, we assume that the eigenstates of $\mathcal{H}$ are non-degenerate. The eigenstates $|n\rangle$ are orthonormal vectors that satisfy

$$\mathcal{H}|n\rangle = E_n|n\rangle, \qquad n \geqslant 0 \tag{3}$$

$$0 = E_0 < E_1 < E_2 < \cdots . \tag{4}$$

Examples where $\lim_{l\to\infty} E_l = \infty$ and $\lim_{l\to\infty} E_l = E^\star < \infty$ will both be of interest. (Generalization of the requirement that $E_0 = 0$ will appear subsequently; inclusion of level degeneracy is discussed in [4].)

We let $E_n = \omega e_n (= \hbar\omega e_n)$, $\omega > 0$ and fixed, and thereby introduce a sequence of dimensionless real numbers $0 = e_0 < e_1 < e_2 < \cdots$. As a preliminary step we first define coherent states of the form

$$|J, \gamma\rangle = N(J)^{-1} \sum_{n=0}^{+\infty} \frac{J^{n/2} \exp(-\mathrm{i} e_n \gamma)}{\sqrt{\rho_n}} |n\rangle \tag{5}$$

where $0 \leqslant J$ and $-\infty < \gamma < +\infty$, and $N(J)$ denotes a normalization chosen so that

$$\langle J, \gamma | J, \gamma\rangle = N(J)^{-2} \sum_{n=0}^{+\infty} \frac{J^n}{\rho_n} \equiv 1. \tag{6}$$

Thus

$$N(J)^2 \equiv \sum_{n=0}^{+\infty} \frac{J^n}{\rho_n}. \tag{7}$$

The domain of allowed J, $0 \leqslant J < R$, is determined by the radius of convergence $R = \overline{\lim}_{n\to\infty} \sqrt[n]{\rho_n}$ in the series defining $N(J)^2$. The radius of convergence may be finite (non-zero) or infinite, depending on the behaviour of ρ_n for large n. These positive constants ρ_n are assumed to arise as the moments of a probability distribution, namely,

$$\rho_n \equiv \int_0^R u^n \rho(u)\, \mathrm{d}u \qquad \rho(u) \geqslant 0. \tag{8}$$

We assume all moments exist: $\rho_0 = 1$ and $\rho_n < +\infty$, for all n.

We take up the resolution of unity next. To that end, we define

$$\int \cdots \mathrm{d}\nu(\gamma) \equiv \lim_{\Gamma\to\infty} \frac{1}{2\Gamma} \int_{-\Gamma}^{\Gamma} \cdots \mathrm{d}\gamma \tag{9}$$

and we consider

$$\begin{aligned} \int |J, \gamma\rangle\langle J, \gamma|\, \mathrm{d}\nu(\gamma) &= \lim_{\Gamma\to\infty} \frac{1}{2\Gamma} \int_{-\Gamma}^{\Gamma} N(J)^{-2} \sum_{m,n=0}^{\infty} \frac{J^{(m+n)/2} \exp(-\mathrm{i}\gamma(e_m - e_n))}{\sqrt{\rho_m \rho_n}} |m\rangle\langle n|\, \mathrm{d}\gamma \\ &= N(J)^{-2} \sum_{n=0}^{\infty} \frac{J^n}{\rho_n} |n\rangle\langle n|. \end{aligned} \tag{10}$$

To complete the derivation of the resolution of unity, we first define

$$k(J) \equiv N(J)^2 \rho(J) \geqslant 0 \qquad 0 \leqslant J < R \tag{11}$$

$$k(J) \equiv \rho(J) \equiv 0 \qquad J > R \tag{12}$$

which trivially implies for the weighted integral over J, $0 \leqslant J < R$,

$$\frac{1}{\rho_n} \int_0^R N(J)^{-2} J^n k(J)\, \mathrm{d}J = 1. \tag{13}$$

Since the sum and integral may be interchanged, as expectations in a general state confirm, we verify that

$$\int_0^R k(J)\,\mathrm{d}J \int |J,\gamma\rangle\langle J,\gamma|\,\mathrm{d}\nu(\gamma) = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{1}. \tag{14}$$

We have thus established the resolution of unity with $\mathrm{d}\mu(J,\gamma) = k(J)\,\mathrm{d}J\,\mathrm{d}\nu(\gamma)$, $0 \leqslant J < R$ and $-\infty < \gamma < +\infty$. Temporal stability follows easily. Since $E_n = \omega e_n$ for all n, we find that

$$\mathrm{e}^{-\mathrm{i}\mathcal{H}t}|J,\gamma\rangle = N(J)^{-1}\sum_{n=0}^{\infty} \frac{J^{n/2}\exp(-\mathrm{i}e_n\gamma - \mathrm{i}E_n t)}{\sqrt{\rho_n}}|n\rangle = |J,\gamma+\omega t\rangle. \tag{15}$$

Thus we have defined coherent states that satisfy the first three requirements, and we find many possible solutions as represented by the various choices of $\rho(u)$ one could select.

To deal with the fourth requirement, we define

$$H(J,\gamma) \equiv \langle J,\gamma|\mathcal{H}|J,\gamma\rangle = N(J)^{-2}\sum_{n=0}^{\infty}\frac{E_n J^n}{\rho_n} = \omega\frac{\sum_{n=0}^{\infty} e_n J^n/\rho_n}{\sum_{n=0}^{\infty} J^n/\rho_n}. \tag{16}$$

To satisfy the last requirement we demand that

$$\sum_{n=0}^{\infty}\frac{e_n J^n}{\rho_n} = J\sum_{n=0}^{\infty}\frac{J^n}{\rho_n}. \tag{17}$$

Recalling that $e_0 = 0$, this requires $e_n = \rho_n/\rho_{n-1}$. Along with $\rho_0 = 1$, this relation leads to

$$\rho_n = e_1 e_2 e_3 \cdots e_n \tag{18}$$

which is the unique criterion for the set of moments $\{\rho_n\}$ to lead to the desired relation that $H(J,\gamma) = \langle J,\gamma|\mathcal{H}|J,\gamma\rangle = \omega J$.

It should be noted that the above factorization formula (18) has already appeared in various previous generalizations of analytical coherent states (see for instance [10] and references therein), specially in works related with q-deformations and q-special functions, where $e_n = (q^n - q^{-n})/(q - q^{-1}) \equiv [n]_q$ and $\rho_n = [n]_q!$. Such a formula has also been proposed by one of the authors [9] for arbitrary increasing sequences of positive numbers e_n. However, its relevance in quantum physics is fully understood in the present context *because* of the relaxing of the analyticity condition on the coherent states.

As a related remark, we also evaluate $i\langle J,\gamma|\mathrm{d}|J,\gamma\rangle$, where

$$\mathrm{d}|J,\gamma\rangle \equiv |J+\mathrm{d}J,\gamma+\mathrm{d}\gamma\rangle - |J,\gamma\rangle.$$

Since $|J,\gamma\rangle$ is a normalized vector, the quantity in question is real. Hence, we are assured that the coefficient of $\mathrm{d}J$ vanishes and we concentrate just on $\mathrm{d}\gamma$. With this restriction,

$$\mathrm{d}|J,\gamma\rangle = (-\mathrm{i}\,\mathrm{d}\gamma)N(J)^{-1}\sum_{n=0}^{\infty}\frac{e_n J^{n/2}\exp(-\mathrm{i}e_n\gamma)}{\sqrt{\rho_n}}|n\rangle \tag{19}$$

and therefore

$$\mathrm{i}\langle J,\gamma|\mathrm{d}|J,\gamma\rangle = \mathrm{d}\gamma\, N(J)^{-2}\sum_{n=0}^{\infty}\frac{e_n J^n}{\rho_n} = J\,\mathrm{d}\gamma \tag{20}$$

as necessary.

As a first example, let us examine the harmonic oscillator for which

$$\mathcal{H}|n\rangle = \omega n|n\rangle \qquad \text{i.e. } e_n = n \tag{21}$$

for all $0 \leqslant n < +\infty$. For this example, the present coherent states,

$$|J, \gamma\rangle = N(J)^{-1} \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-in\gamma}}{\sqrt{n!}} |n\rangle \qquad N(J)^2 = e^J \tag{22}$$

are just the usual canonical coherent states

$$|z\rangle \equiv e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \tag{23}$$

with $z \equiv \sqrt{J} e^{-i\gamma}$. We note further, in this case, that the radius of convergence $R = \infty$ and that $\rho(J) = e^{-J}$ and hence $k(J) = 1$. Furthermore, in this case, due to the nature of the spectrum, it suffices to choose $d\nu(\gamma) = d\gamma/2\pi$ for $-\pi \leqslant \gamma < \pi$, which already provides the necessary projection of $m \neq n$ terms in the resolution of the unity.

Thus the present four requirements have uniquely led to the usual coherent states for the harmonic oscillator. Additional sets of coherent states appropriate to Hamiltonians for which $\lim_{n\to\infty} E_n = \infty$ will be treated elsewhere [7].

Let us next consider an example for which $\lim_{n\to\infty} E_n = E^\star < \infty$. As our example, we choose a Coulomb-like spectrum [8] and consider the unit operator as the projector onto the subspace generated by the discrete spectrum only.

$$E_{n-1} \equiv \omega \left[1 - \frac{1}{n^2}\right] \qquad n \geqslant 1 \tag{24}$$

$$e_{n-1} = 1 - \frac{1}{n^2} = \frac{(n+1)(n-1)}{n^2} \tag{25}$$

with a constant added so that $E_0 = 0$. In this case

$$\rho_n = e_1 e_2 \cdots e_n = \frac{3}{4} \times \frac{8}{9} \cdots \frac{n(n+2)}{(n+1)^2} = \frac{1}{2}\frac{(n+2)}{(n+1)}. \tag{26}$$

Therefore, $\lim e_n = 1$ and $\lim \rho_n = \frac{1}{2}$ as $n \to \infty$. This means the radius of convergence in the present case is $R = 1$, i.e. $0 \leqslant J < 1$. In addition,

$$N(J)^2 = \frac{2}{J}\left[\frac{1}{1-J} + \frac{\ln(1-J)}{J}\right] \tag{27}$$

$$\rho(u) = \frac{1}{2}\left[1 + \delta(u - 1^-)\right] \qquad 0 \leqslant u < 1 \tag{28}$$

and we observe that

$$\int_0^1 N(J)^{-2} J^n [N(J)^2 \rho(J)]\, dJ = \rho_n \tag{29}$$

as required.

Thus we have obtained a set of coherent states of the form

$$|J, \gamma\rangle = N(J)^{-1} \sum_{n=0}^{\infty} \sqrt{\frac{2n+2}{n+2}}\, J^{n/2} \exp\left(-i\left[1 - \frac{1}{(n+1)^2}\right]\gamma\right) |n\rangle. \tag{30}$$

It follows that

$$e^{-i\mathcal{H}t} |J, \gamma\rangle = |J, \gamma + \omega t\rangle \tag{31}$$

as required, and also that

$$\begin{aligned} \langle J, \gamma | \mathcal{H} | J, \gamma\rangle &= \omega N(J)^{-2} \sum \left(\frac{2n+2}{n+2}\right) J^n \left[1 - \frac{1}{(n+1)^2}\right] \\ &\equiv \omega \sum \left(\frac{2n+2}{n+2}\right) \frac{n(n+2)}{(n+1)^2} J^n \left(\sum \left(\frac{2n+2}{n+2}\right) J^n\right)^{-1} = \omega J \end{aligned} \tag{32}$$

which establishes that the coherent states for this Coulomb-like model satisfy our four requirements.

Remark. It would be of interest to extend these states to degenerate levels, in the manner of [4], and compare the resultant coherent states for the (bound state portion of the) hydrogen atom with coherent states having alternative definitions.

If one does not choose to set $E_0 = 0$, then it is possible to consider

$$e^{-i\mathcal{H}t}|J,\gamma\rangle = \exp(-iE_0 t)\exp(-i(\mathcal{H}-E_0)t)\,|J,\gamma\rangle = \exp(-iE_0 t)\,|J,\gamma+\omega t\rangle \tag{33}$$

which although not identical to the third requirement would constitute a modest generalization.

3. Coherent states for continuum dynamics

Let $\mathcal{H} > 0$ be a Hamiltonian with a non-degenerate continuous spectrum, and let $|E\rangle$ denote the formal (delta-function normalized) states for which

$$\mathcal{H}|E\rangle = \omega E|E\rangle \qquad 0 < E < \bar{E} \tag{34}$$

where $\bar{E} < \infty$ or $\bar{E} = \infty$ are both of interest. In the present case we abusively denote $|J,\gamma\rangle$ by $|s(J),\gamma\rangle$, where $s = s(J) > 0$ is to be determined, and we set

$$|s,\gamma\rangle \equiv M(s)^{-1}\int_0^{\bar{E}} \frac{s^E e^{-i\gamma E}}{f(E)}|E\rangle\,dE \tag{35}$$

where $f(E)$ is specified below. Insisting that $\langle s,\gamma|s,\gamma\rangle = 1$ leads to

$$M(s)^2 = \int_0^{\bar{E}} \frac{s^{2E}}{f(E)^2}\,dE \tag{36}$$

for $0 \leqslant s < S$, where $S = \overline{\lim}\, s$ for all s for which $M(s)^2 < \infty$. Either $S < \infty$ or $S = \infty$.

We next construct a resolution of identity from these states. To that end consider first

$$\begin{aligned}
&\int_{-\infty}^{+\infty} |s,\gamma\rangle\langle s,\gamma|\,\frac{d\gamma}{2\pi} \\
&\quad = M(s)^{-2}\int_{-\infty}^{+\infty}\int_0^{\bar{E}}\int_0^{\bar{E}} \frac{s^{E+E'}\exp\left(-i\gamma(E-E')\right)}{f(E)f(E')}|E\rangle\langle E'|\,\frac{d\gamma}{2\pi}\,dE\,dE' \\
&\quad = M(s)^{-2}\int_0^{\bar{E}} \frac{s^{2E}}{f(E)^2}|E\rangle\langle E|\,dE.
\end{aligned} \tag{37}$$

Next we introduce a non-negative weight function $\sigma(s) \geqslant 0$ (for example, $\exp(-s^\alpha)$, $0 < \alpha$) such that

$$\int_0^S s^{2E}\sigma(s)\,ds \equiv f(E)^2 \tag{38}$$

for non-negative $\sigma(s) \geqslant 0$. This equation implicitly defines allowed $f(E)$, and $0 < f(E) < \infty$, for $0 < E < \bar{E}$. With $d\mu(s,\gamma) \equiv (1/2\pi)M(s)^2\sigma(s)\,ds\,d\gamma$ we have

$$\begin{aligned}
\int |s,\gamma\rangle\langle s,\gamma|\,d\mu(s,\gamma) &= \int_0^S ds\, M(s)^2\sigma(s)\int_{-\infty}^{+\infty}|s,\gamma\rangle\langle s,\gamma|\,\frac{d\gamma}{2\pi} \\
&= \int_0^{\bar{E}} |E\rangle\langle E|\,dE = \mathbb{1}
\end{aligned} \tag{39}$$

as desired.

Temporal stability follows immediately, since

$$e^{-i\mathcal{H}t}|s,\gamma\rangle = M(s)^{-1}\int_0^{\bar{E}} \frac{s^E e^{-iE(\gamma+\omega t)}}{f(E)}|E\rangle \, dE = |s,\gamma+\omega t\rangle. \tag{40}$$

To generate such coherent states, it is easiest to start with $\sigma(s)$, define $f(E)$, and then find the normalization factor $M(s)^2$. In this way, whole families of coherent states can be generated. There remains only to find the consequences of imposing the fourth requirement.

We consider

$$H(s) \equiv \langle s,\gamma|\mathcal{H}|s,\gamma\rangle = M(s)^{-2}\int_0^{\bar{E}} \frac{Es^{2E}}{f(E)^2}\, dE = s\frac{\partial}{\partial s}\ln M(s). \tag{41}$$

If we set $Y(s) \equiv \ln M(s)$, then

$$H(s) = s\frac{\partial}{\partial s}Y(s). \tag{42}$$

In general, $H(s) \neq \omega s$ as the fourth requirement wants, but that only means we have chosen the wrong variable. Therefore, let us set

$$\omega J \equiv H(s) \tag{43}$$

and assume this relation is uniquely invertible, leading to

$$s = s(J). \tag{44}$$

The condition for invertibility is that

$$H'(s) = (sY'(s))' > 0 \tag{45}$$

for all s (it is also possible to have $H'(s) < 0$). For example, if $Y(s) = s^\alpha$, $\alpha > 0$, then $H'(s) = \alpha^2 s^{\alpha-1} > 0$ as required. Let us assume we can find $s = s(J)$ such that $H(s) = H(s(J)) = \omega J$, which amounts to solve the following integral equation (equivalent to (41)) for a given admissible $f(E)$:

$$\exp\left(-2\omega\int \ln s(J)\, dJ\right) = \int_0^{\bar{E}} \frac{s(J)^{2(E-\omega J)}}{f(E)^2}\, dE. \tag{46}$$

We now recapitulate the specification of the coherent states for a system with a non-degenerate, continuum spectrum $E > 0$. We define

$$|J,\gamma\rangle \equiv P(J)^{-1}\int_0^{\bar{E}} \frac{s(J)^E e^{-i\gamma E}}{f(E)}|E\rangle\, dE \tag{47}$$

$$P(J)^2 \equiv M(s(J))^2 = \int_0^{\bar{E}} \frac{s(J)^{2E}}{f(E)^2}\, dE \tag{48}$$

and observe with $\tau(J)\, dJ \equiv \sigma(s(J))\, ds(J)$ and $S = s(\bar{J})$, i.e. $\bar{J} = H(S)$, that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\gamma \int_0^{\bar{J}} |J,\gamma\rangle\langle J,\gamma| P(J)^2 \tau(J)\, dJ = \int_0^{\bar{E}} |E\rangle\langle E|\, dE = \mathbb{1} \tag{49}$$

as desired. Temporal stability follows as before since

$$e^{-i\mathcal{H}t}|J,\gamma\rangle = P(J)^{-1}\int_0^{\bar{E}} \frac{s(J)^E e^{-iE(\gamma+\omega t)}}{f(E)}|E\rangle\, dE = |J,\gamma+\omega t\rangle. \tag{50}$$

Finally, we observe that

$$\langle J,\gamma|\mathcal{H}|J,\gamma\rangle = \omega\frac{\int_0^{\bar{E}}\left(Es(J)^{2E}/f(E)^2\right) dE}{\int_0^{\bar{E}}\left(s(J)^{2E}/f(E)^2\right) dE} = \omega J \tag{51}$$

as required. Conditions for these relations to hold are that $\tau(J) \geqslant 0$ and $H'(s) > 0$ (or < 0), which have already been assumed.

There are some similarities and some differences in the cases for a discrete or continuum spectrum. For the discrete case the dependence on J through $J^{n/2}$ is 'natural' while the dependence on γ through $\exp(-\mathrm{i}e_n\gamma)$ is 'unnatural'. (Un)naturalness is here understood by comparison with the standard case. In the same sense, in the continuum case, the dependence on J through $s(J)^E$ is unnatural while the dependence on γ through $\mathrm{e}^{-\mathrm{i}E\gamma}$ is natural. Only the harmonic oscillator has a natural dependence on both J and γ, while that is not the case in general. Another important difference in the two cases is that for a discrete spectrum one needs

$$\int (\cdot)\, \mathrm{d}\nu(\gamma) \equiv \lim_{\Gamma\to\infty} \frac{1}{2\Gamma} \int_{-\Gamma}^{\Gamma} (\cdot)\, \mathrm{d}\gamma \tag{52}$$

which denotes an *average* over a wide, flat distribution. For the continuous spectrum, on the other hand, one uses

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} (\cdot)\, \mathrm{d}\gamma \tag{53}$$

which illustrates a relative scale factor in the integration measure for γ of 0 (or ∞) between the two measures. Finally, we observe in the discrete case that the coherent states were determined uniquely by the four requirements, while for the continuous case some freedom remains, specifically in choosing $\sigma(u)$ so that, ultimately, $J = H(s)/\omega$ admits a unique inverse. Of course, this arbitrariness arises because we permit ourselves a final change of parameters $s \to J$.

Suppose, like the discrete case, we started straight away with the proposal that

$$|J, \gamma\rangle\!\rangle = Q(J)^{-1} \int_0^\infty \frac{J^{E/2}\mathrm{e}^{-\mathrm{i}\gamma E}}{f(E)} |E\rangle\, \mathrm{d}E \tag{54}$$

and insisted that

$$\langle\!\langle J, \gamma|\mathcal{H}|J, \gamma\rangle\!\rangle = \omega \frac{\int \left(EJ^E/f(E)^2\right) \mathrm{d}E}{\int \left(J^E/f(E)^2\right) \mathrm{d}E} = \omega J. \tag{55}$$

For this relation to hold it is necessary that

$$J \frac{\partial}{\partial J} \ln \int J^E g(E)\, \mathrm{d}E = J \tag{56}$$

where $g(E) = 1/f(E)^2$. This differential equation admits the solution

$$\int J^E g(E)\, \mathrm{d}E = \mathrm{e}^J \tag{57}$$

from which it follows that

$$g(E) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta(E - n) = \frac{1}{f(E)^2}. \tag{58}$$

Although we have obtained a unique result in this fashion, we deem it to be unacceptable, because it leads to states $|J, \gamma\rangle\!\rangle$ that do not even span the Hilbert space let alone fulfill a resolution of unity, and are poorly defined as well.

4. Coherent states for discrete and continuous dynamics

In sections 2 and 3 we have developed coherent states separately for systems with discrete spectra and for systems with continuous spectra. Now we wish to join the two in order to be able to discuss the case of a Hamiltonian $\mathcal{H}$ that possesses both spectral types. We do so in a relatively straightforward manner using states of each kind for their respective subspaces. We use the original notation $|J, \gamma\rangle$ for *discrete-spectrum coherent states*, and, for purposes of this section, we use the notation $|K, \delta\rangle$ for *continuous-spectrum coherent states* (replacing the former $|J, \gamma\rangle$ notation). As coherent states for the combined system we choose the *unnormalized* states

$$|J, \gamma; K, \delta; \phi\rangle = f(K, \delta)|J, \gamma\rangle + \mathrm{e}^{-\mathrm{i}\phi} g(J, \gamma)|K, \delta\rangle \tag{59}$$

where f and g are scalar functions to be determined. Continuity of the combined coherent states follows from continuity of the separate states and of the functions f and g, which we now assume.

The resolution of unity, namely

$$\int |J, \gamma; K, \delta; \phi\rangle\langle J, \gamma; K, \delta; \phi| \,\mathrm{d}\lambda(J, \gamma; K, \delta; \phi) = \mathbb{1} \tag{60}$$

which is the direct sum of $\mathbb{1}_D$ (discrete) and $\mathbb{1}_C$ (continuous) that apply in the separate spaces, may be achieved as follows. There are several ingredients, specifically three, that need to be satisfied. They are

$$\int |f(K, \delta)|^2 |J, \gamma\rangle\langle J, \gamma| \,\mathrm{d}\lambda = \mathbb{1}_D \tag{61}$$

$$\int |g(J, \gamma)|^2 |K, \delta\rangle\langle K, \delta| \,\mathrm{d}\lambda = \mathbb{1}_C \tag{62}$$

$$\int \mathrm{e}^{\mathrm{i}\phi} \int g(J, \gamma)^\star f(K, \delta) |J, \gamma\rangle\langle K, \delta| \,\mathrm{d}\lambda = 0 \tag{63}$$

as well as the adjoint of the last relation. It is natural to choose

$$\mathrm{d}\lambda(J, \gamma; K, \delta; \phi) = \mathrm{d}\mu_D(J, \gamma) \,\mathrm{d}\mu_C(K, \delta) \,\mathrm{d}\phi/2\pi \tag{64}$$

as a product measure. In that case, the three conditions take the form

$$\int |f(K, \delta)|^2 \,\mathrm{d}\mu_C(K, \delta) = 1 \tag{65}$$

$$\int |g(J, \gamma)|^2 \,\mathrm{d}\mu_D(J, \gamma) = 1 \tag{66}$$

while integration over ϕ, $0 \leqslant \phi < 2\pi$, eliminates the unwanted off-diagonal terms.

In particular, let us choose

$$g(J, \gamma) \equiv N_g \exp\left(-J^2\right) \tag{67}$$

where the factor N_g is chosen to ensure that

$$N_g^2 \int \exp\left(-2J^2\right) \,\mathrm{d}\mu_D(J, \gamma) = 1. \tag{68}$$

Further, we set

$$f(K, \delta) = N_f \exp\left(-K^2 - \delta^2\right) \tag{69}$$

and choose N_f so that

$$N_f^2 \int \exp\left(-2K^2 - 2\delta^2\right) \, d\mu_C(K, \delta) = 1. \tag{70}$$

Thus we have ensured the existence of a resolution in the combined space.

To deal with temporal stability we assume that $0 \leqslant \mathcal{H}_D \leqslant \Omega$, as in the Coulomb-like problem of section 2, and, as a consequence, $\Omega < \mathcal{H}_C$. This meshing of the two spectra means that we must appeal to utilizing the additional phase factor. Thus we find

$$\begin{aligned} e^{-i\mathcal{H}t}|J, \gamma; K, \delta; \phi\rangle &= f(K, \delta)|J, \gamma + \omega t\rangle + e^{-i(\phi+\Omega t)} g(J, \gamma)|K, \delta + \omega t\rangle \\ &= |J, \gamma + \omega t; K, \delta + \omega t; \phi + \Omega t\rangle \end{aligned} \tag{71}$$

which remains a coherent state and exhibits temporal stability.

Finally, we note that the action identity seems to be difficult to obtain with the combined coherent states, at least in any straightforward fashion, and we do not pursue that issue further.

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